UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH3401**

ASSESSMENT PATTERN : **MATH3401A**

MODULE NAME : **Mathematical Methods 5**

DATE : **12-May-10**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2009/10-MATH3401A1-001-EXAM-34

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Use the method of variation of parameters to show that the solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = f(x), \quad y(0) = \frac{dy}{dx}(0) = 0,$$

is

$$y(x) = \int_0^x (x-u)e^{x-u} f(u)\, du.$$

(b) (i) Determine a function $f(t)$ of the complex variable t and contours C_1 and C_2 in the complex t-plane so that

$$\int_{C_i} e^{xt} f(t)\, dt, \quad i = 1, 2,$$

are non trivial solutions of the differential equation

$$x\frac{d^2y}{dx^2} + (n+1-x)\frac{dy}{dx} - ny = 0, \quad n = 1, 2, 3, 4, \cdots.$$

(ii) Show that the solution $y(x)$ to this equation which is bounded as $x \to \infty$ is singular at $x = 0$ and show that, in this case,

$$x^n y(x) = y(1).$$

2. (a) Consider the pair of first order differential equations

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y),$$

for the functions $x(t)$ and $y(t)$.

(i) Define the phase plane for this system and explain why periodic solutions correspond to closed trajectories in the phase plane.

(ii) By considering the integral

$$\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dx \, dy,$$

over a region D of the phase plane, show that periodic solutions are impossible in regions of the plane where

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

does not change sign.

(b) Consider the second order differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x - x^3 = 0,$$

for $x(t)$.

(i) Show this can be written as the first order system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -y - x + x^3.$$

(ii) Use the result of part (a) to show that there are no periodic solutions to this equation.

(iii) Examine the phase plane of this equation, identifying the vertical and horizontal nullclines, the singular points and their nature.

(iv) Sketch the phase trajectories.

3. A variant of the Van der Pol equation is

$$\frac{d^2x}{dt^2} + \varepsilon(x^n - 1)\frac{dx}{dt} + x = 0,$$

for $x(t)$, $\varepsilon > 0$, $n = 1, 2, 3, 4, \cdots$. The usual Van der Pol equation is given by the case $n = 2$.

(a) If $\varepsilon << 1$, then look for periodic solutions with amplitude a using the expansions

$$x(t) = a\cos(\theta) + \varepsilon x_1(\theta) + \cdots, \quad \theta = (1 + \varepsilon n_1 + \cdots)t,$$

and show that

$$n_1 = 0, \quad a = \left(\frac{2^{n-1}(n+2)[(n/2)!]^2}{n!}\right)^{\frac{1}{n}}$$

if n is even, but that no such solution is possible if n is odd.

You may quote the following result

$$\int_0^{2\pi} \cos^n\theta \, d\theta = \begin{cases} 0 & n = 1, 3, 5, \cdots \\ \frac{2\pi\, n!}{2^n[(n/2)!]^2} & n = 0, 2, 4, \cdots \end{cases}.$$

(b) If $\varepsilon >> 1$, use the Leinhard transformation

$$y(t) = \frac{dx}{dt} + \varepsilon F(x), \quad F(x) = \frac{x^{n+1}}{n+1} - x,$$

to write the equation in the form

$$\frac{dx}{dt} = y - \varepsilon F(x), \quad \frac{dy}{dt} = -x.$$

By identifying possible closed trajectories in the Leinhard (x-y) plane, show that, if n is even, periodic solutions are possible with period

$$2\,\varepsilon \int_1^{\alpha} \frac{u^n - 1}{u}\, du,$$

with α the maximum value of x attained in the solution and given by the positive root of

$$\alpha^{n+1} - (n+1)\alpha = n.$$

Show that no similar solutions are possible for odd values of n.

4. Consider the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x = \varepsilon f\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right),$$

for $x(t)$, with $\varepsilon << 1$.

(a) If $T = \varepsilon t$ is a slow time, show using the method of multiple scales that

$$x(t) \sim A(T)\sin(t + \Phi(T)),$$

where

$$\frac{\mathrm{d}A}{\mathrm{d}T} = \frac{1}{2\pi}\int_0^{2\pi} \cos\chi f(A\cos\chi)\ \mathrm{d}\chi,$$

$$A\frac{\mathrm{d}\Phi}{\mathrm{d}T} = -\frac{1}{2\pi}\int_0^{2\pi} \sin\chi f(A\cos\chi)\ \mathrm{d}\chi.$$

(b) Deduce that Φ is independent of T.

(c) If $f(u) = u - \alpha u^n$, with n an even integer, show that $A(T)$ grows exponentially and no limit cycle solution of this type exists.

(d) If $f(u) = u - \alpha u^n$, with n an odd integer and $\alpha > 0$, show that the solution approaches a limit cycle solution

$$x(t) = A_p \sin(t + \phi),$$

where A_p should be found. *It is not necessary to explicitly solve a differential equation to deduce this result.*

You may quote the following result

$$\int_0^{2\pi} \cos^n\theta\ \mathrm{d}\theta = \begin{cases} 0 & n = 1, 3, 5, \cdots \\ \frac{2\pi\ n!}{2^n[(n/2)!]^2} & n = 0, 2, 4, \cdots \end{cases}.$$

5. (a) (i) State without proof a form of Watson's Lemma.

(ii) Throughout the interval $a \le t \le b$, the function $f(t)$ is continuous and the function $\phi(t)$ is decreasing with $\phi'(a) \neq 0$. Show, using Watson's Lemma or otherwise, that

$$\int_a^b e^{x\phi(t)} f(t)\, dt \sim -\frac{e^{x\phi(a)} f(a)}{x\phi'(a)}.$$

How is this result modified if $\phi(t)$ is increasing, with $\phi'(b) \neq 0$?

(iii) Using Watson's Lemma or otherwise, show that, if $\phi'(a) = 0$, but $\phi''(a) \neq 0$,

$$\int_a^b e^{x\phi(t)} f(t)\, dt \sim e^{x\phi(a)} f(a) \sqrt{\frac{\pi}{2|\phi''(a)|x}}.$$

(b) Use the results above to derive the following as $x \to \infty$,

(i)

$$\int_0^\infty e^{-xt^2} \sin t\, dt \sim \frac{1}{2x},$$

(ii)

$$\int_{-\pi/2}^{\pi/2} (t+2) e^{-x\cos t}\, dt \sim \frac{4}{x},$$

(iii)

$$\int_0^\infty e^{(x^2t^2 - t^4)} g(t/x)\, dt \sim \sqrt{\frac{\pi}{2}} \frac{e^{\frac{1}{4}x^4}}{x} g(1/\sqrt{2})$$